THE WEAK TYPE $(1,1)$ BOUNDS FOR THE MAXIMAL FUNCTION ASSOCIATED TO CUBES GROW TO INFINITY WITH THE DIMENSION

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Abstract. Let M_d be the centered Hardy-Littlewood maximal function associated to cubes in $\mathbb{R}^d$ with Lebesgue measure, and let c_d denote the lowest constant appearing in the weak type (1,1) inequality satisfied by M_d. We show that $c_d \to \infty$ as $d \to \infty$, thus answering, for the case of cubes, a long standing open question of E. M. Stein and J. O. Strömberg.

1. Introduction and result.

By a cube $Q(x,r)$ we mean a closed ℓ_∞ ball of radius r and center x in $\mathbb{R}^d$, that is, a closed cube centered at x, with sides parallel to the coordinate axes, and sidelength $2r$. Let M_d be the centered maximal function

$$M_d f(x) := \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| dy \tag{1}$$

associated to cubes and Lebesgue measure in $\mathbb{R}^d$. A fundamental feature of the Hardy-Littlewood maximal function M is that it satisfies the weak-type $(1,1)$ inequality: There exists a constant $c > 0$ such that for all $\alpha > 0$ and all $f \in L^1$,

$$\alpha |\{Mf \geq \alpha\}| \leq c \|f\|_1. \tag{2}$$

Denote by c_d the best (i.e. lowest) constant satisfying (2) in $\mathbb{R}^d$. We prove that $c_d \uparrow \infty$ as $d \uparrow \infty$.

Theorem 1.1. *Fix $T > 0$. Then there exists a $D = D(T)$ such that for every dimension $d \geq D$, $c_d \geq T$.*

It is well known that given $1 < p \leq \infty$, there exists a constant c_p such that for all $f \in L^p(\mathbb{R}^d)$, $\|Mf\|_p \leq c_p \|f\|_p$. When $p = \infty$, trivially $c_p = 1$ in every dimension, since averages never exceed a supremum. Dimension independent estimates are useful whenever one is interested in extending results from the finite dimensional to the infinite dimensional setting. For the maximal function associated to euclidean balls, E. M. Stein showed that one can take c_p to be independent of d ([St1], [St2], see also [St3]). Stein's result was generalized to the maximal function defined using balls given by arbitrary norms by J. Bourgain ([Bou1], [Bou2], [Bou3]) and A. Carbery ([C]) when $p > 3/2$. Given $1 \leq q < \infty$, the ℓ_q balls

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are defined using the norm $\|x\|_q := (|x_1|^q + |x_2|^q + \cdots + |x_d|^q)^{1/q}$, and the ℓ_∞ balls, using $\|x\|_\infty := \max_{1\le i\le d}\{|x_1|, |x_2|, \ldots, |x_d|\}$. For ℓ_q balls, $1 \le q < \infty$, D. Müller [Mu] showed that uniform bounds again hold for every $p > 1$. With respect to weak type bounds, in [StSt] E. M. Stein and J. O. Strömberg proved that the best constants in the weak type (1,1) inequality satisfied by the maximal function associated to arbitrary balls grow at most like $O(d \log d)$, while if the balls are euclidean, then the best constants grow at most like $O(d)$. They also asked if uniform bounds could be found. Theorem 1.1 shows that in the case of cubes the answer is negative. If the d-dimensional Lebesgue measures are replaced by a sequence of finite, absolutely continuous radial measures with decreasing densities (such as, for instance, the standard Gaussian measures) then best constants grow exponentially with d, cf. [A2].

In recent years evidence has been mounting to the effect that not only weak type $(1, 1)$ inequalities are formally stronger than strong (p, p) inequalities for $1 < p < \infty$ (since the latter are implied by the former via interpolation) but they are also stronger in a substantial way, meaning that the strong type may hold for all $p > 1$ while the weak type $(1, 1)$ may fail. This is the case, for instance, with the uncentered maximal function associated to the standard gaussian measure and euclidean balls. It is shown in [Sj] that this maximal function is not of weak type $(1, 1)$, while it is strong (p, p) for all $p > 1$, cf. [FSSjU] (for cubes the strong (p, p) type follows from a more general result in [CF, cf. Theorem 1]). Theorem 1.1 may represent another instance of this phenomenon, with respect to uniform bounds in d. However, it is not known for cubes whether uniform bounds hold when $1 < p \le 3/2$ (it is suggested in [Mu] that the answer may be negative, and conjectured in [ACP] that the answer is positive).

Before presenting the proof, we make some comments on the method of discretization for weak type $(1, 1)$ inequalities. It consists in replacing L^1 functions by finite sums of Dirac deltas. This leads to elementary arguments of a combinatorial nature. The fact that one can get lower bounds for c_d using Dirac deltas instead of functions is obvious, by mollification. And this all we need here.

We mention for completeness that considering Dirac deltas also suffices to give upper bounds, as shown by M. de Guzmán, see [Gu, Theorem 4.1.1]. Furthermore, M. Trinidad Menárguez and F. Soria proved that discretizing does not alter constants, cf. [MS, Theorem 1], so it can be used to study the precise values of c_d. This method was utilized, for instance, in [A], were it was shown that $37/24 \le c_1 \le \frac{9+\sqrt{41}}{8}$, thereby refuting the conjecture that $c_1 = 3/2$ (cf. [BH, Problem 7.74 c]) and showing that $c_1 < 2$, which is the best constant in the uncentered case. Discretization was also used in [Me1] and [Me2], where the exact value of $c_1 = \frac{11+\sqrt{61}}{12}$ was found. No best constants are known for dimensions larger than one.

Let us point out that the configuration of Dirac deltas we will utilize had previously been considered in [MS, Theorem 6], for the same purpose of bounding c_d from below. It is shown there that $c_d \ge \left(\frac{1+2^{1/d}}{2}\right)^d$. Trivially, these lower bounds are bounded above by 2 (in fact they start with $c_1 = 3/2$ and approach $\sqrt{2}$ as $d \to \infty$) so they do not decide the issue of whether

there is a uniform upper bound. Actually, it is proven in [AV, Theorem 2] that $c_{d+1} \geq c_d$; hence, the best bounds are nondecreasing in d.

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2. Proof.

Given a locally finite measure ν, the maximal function $M_d\nu$ is defined by

$$M_d\nu(x) := \sup_{r>0} \frac{\nu Q(x,r)}{|Q(x,r)|}. \tag{3}$$

For notational simplicity we start considering the infinite measure μ^d in $\mathbb{R}^d$ obtained by placing one Dirac delta at each point of the integer lattice $\mathbb{Z}^d$. The finite measure exhibiting a lower bound for c_d will then be obtained by restricting μ^d to a sufficiently large cube. Note that $\mu^d = \mu^1 \times \mu^1 \times \cdots \times \mu^1$. At first, we will work within the unit cube $[0,1]^d$ only.

Given $u \in (0, 1/2)$ and an interval $I \subset \mathbb{R}$, call $y \in I$ centered at level u (more briefly, centered) if it belongs to the closed subinterval with the same center and length $(1-2u)|I|$, and off center (at level u) otherwise. In particular, for $I = [0,1]$ the centered points are those in $[u, 1-u]$. The role of u in the proof is to serve as a discrete parameter, used to describe which cubes should be considered when estimating the value of the maximal function at a given point.

We shall see that the maximal function is large on the set $E_u \subset [0,1]^d$ of points $x = (x_1, \dots, x_d)$ with many centered coordinates, where "large" is determined by a fixed $t >> 1$, and "many" means more than $(1-2u)d + t\sqrt{d(2u)(1-2u)}$. Since $t\sqrt{d(2u)(1-2u)}$ amounts to t standard deviations of a binomially distributed random variable with parameters d and $2u$, the Central Limit Theorem allows us to bound $|E_u|$ from below independently of u, provided d is large enough (we mention that a similar argument can be carried out on the set of points with many uncentered coordinates). For a fixed u the measure of $|E_u|$ as $d \to \infty$ turns out to be too small, since we are $t >> 1$ standard deviations away from the mean. On the other hand, estimates for the size of $M\mu^d$ worsen when we have roughly $(1-2u)d$ centered coordinates. Changing the value of u by discrete steps and taking the union of many E_u's, we obtain a sufficiently large set over which $M_d\nu$ can be shown to take high values (unlike u, the value of t is fixed throughout the argument, so dependency on t is not indicated in the notation). In order to control the intersections of different E_u's, it is useful to also bound from below the number of uncentered coordinates.

Fix $t >> 1$. The assumption that t is very large will be used without further mention, in a few estimates of the type $\log t << t^{1/3}$. But we emphasize that the value of t remains unchanged throughout the proof; in particular, it does not approach ∞ as $d \to \infty$. So we will assume, again without further mention, that expressions such as $t/\sqrt{d}$ are as small as needed each time they appear.

Let $i = 1, \ldots, k$. Define, for each $u \in (0, 1/4]$,

$$(4) \qquad E_u := \{x \in [0,1]^d : \text{ for at most } k \le 2ud - t\sqrt{d(2u)(1-2u)} \text{ coordinates } j_1, \ldots, j_k,$$
$$x_{j_i} \in [0, u) \cup (1-u, 1]\}$$

and

$$(5) \qquad E^u := \{x \in E_u : \text{ for at least } k > 2ud - (t + t^{-1} 4 \log t)\sqrt{d(2u)(1-2u)} \text{ coordinates},$$
$$x_{j_i} \in [0, u) \cup (1-u, 1]\}.$$

We shall call E^u a *block*, the block associated to u. If instead of taking the union over many values of k we consider the case of exactly k off center coordinates, we call this a $d-k$ *set*. Each $d-k$ set has as its components what we call $d-k$ *faces*. These "faces" are associated to the $d-k$ dimensional faces of $[0,1]^d$, and are obtained from them via a thickening by u in the directions of the off center coordinates, and a shrinking to $1-2u$ in the directions of the centered coordinates. The $d-k$ faces are determined by specifying which k coordinates are off center, and for each such coordinate x_j, whether it is near 0 (i.e., $x_j \in [0, u)$), or near 1 (i.e., $x_j \in (1-u, 1]$). Of course, the number of $d-k$ dimensional faces of $[0,1]^d$, or equivalently, the number of $d-k$ faces in a $d-k$ set, is $2^k\binom{d}{k}$. Observe that within a block, different $d-k$ sets are disjoint. This is clear under a probabilistic interpretation: Call being off center a success, and note that different $d-k$ sets correspond to different number of successes. Likewise, within a $d-k$ set, different $d-k$ faces are disjoint. It follows that (for a fixed u) $d-k$ faces are disjoint with $d-j$ faces. In short, all faces defined by u are disjoint. Additionally, $d-k$ faces are products of one dimensional intervals, and they are all congruent. These facts will be used when we estimate the intersection between blocks.

Finally, we shall set $0 < u = jt^{-4/3} \le 1/4$, where j is a positive integer. So actually we will always have $t^{-4/3} \le u \le 1/4$. Neither bound is particularly important, we could choose instead $1/8 \le u \le 3/8$, for instance. What really matters is that we are considering around $t^{4/3}$ different values of u, and any such two values are at a distance of at least $t^{-4/3}$.

The proof now proceeds by showing, first, that for each $u = t^{-4/3}, 2t^{-4/3}, 3t^{-4/3} \cdots \le 1/4$, $M\mu^d$ is large on E^u. Second, by estimating the size of E^u. Third, by showing that thanks to the minimal separation $t^{-4/3}$, different E^u's have very small intersection. And fourth, by taking the union of the sets E^u. Up to here, the argument is carried inside $[0,1]^d$. To complete the proof we replace μ^d by a finite measure, and apply the estimates obtained within $[0,1]^d$ to several translates of it.

Claim 1: Fix $u \in (0, 1/4]$. For every $\varepsilon_0 > 0$ there exists a D such that if $d \ge D$, then $E_u \subset \{M\mu^d > e^{-\varepsilon_0} e^{t^2/2}\}$.

Proof. Let $x_j \in [u, 1-u]$. Given any integer $s > 0$, $\mu^1[x_j - (s-u), x_j + s - u] = 2s$. Now suppose $y \in [0,1]$ is off center, say for instance $y > 1-u$. Shifting the preceding interval to the right by $y - x_j$ (so it is centered at y) loses at most one Dirac delta on the left. Thus, $\mu^1([y - (s-u), y + s - u]) \ge 2s - 1$. Let $[w]$ denote the integer part of $w > 0$. Suppose

$x \in [0,1]^d$ has r off center and $d-r$ centered coordinates, where $r \le 2ud - t\sqrt{d(2u)(1-2u)}$. Then for every $s = 1, 2, 3, \ldots,$

$$M_d\mu^d(x) \ge \frac{(2s)^{d-r}(2s-1)^r}{\left(2\,(s-u)\right)^d} = \frac{\left(1-\frac{1}{2s}\right)^r}{\left(1-\frac{u}{s}\right)^d} \ge \frac{\left(1-\frac{1}{2s}\right)^{\left[2ud - t\sqrt{d(2u)(1-2u)}\right]}}{\left(1-\frac{u}{s}\right)^d}. \tag{6}$$

The next step consists in showing that for all sufficiently large d and some suitably chosen s we have

$$\frac{\left(1-\frac{1}{2s}\right)^{2u - t\sqrt{2u(1-2u)}/\sqrt{d}}}{1-\frac{u}{s}} \ge 1 + \frac{t^2}{2d} + O\left(\frac{1}{d^{3/2}}\right). \tag{7}$$

Set $f(s) := \left(1-\frac{1}{2s}\right)^y / \left(1-\frac{u}{s}\right)$, where $y \in (4u/3, 2u)$. An elementary calculus argument shows that $f(s)$ is maximized over $s > 1$ when

$$s = \frac{u(1-y)}{2u-y} =: s_0, \tag{8}$$

and this is the only critical point, so f decreases as we move away from s_0. In particular, $f(s_0) \ge f([s_0]) \ge f(s_0 - 1)$. Setting $y = 2u - t\sqrt{2u(1-2u)}/\sqrt{d}$ we obtain $s_0 = (\sqrt{du(1-2u)} + ut\sqrt{2})/(t\sqrt{2})$. Since $f([s_0]) \ge f(s_0-1)$ and $[yd] \le yd$, it is possible to use in (6) the more convenient values $s_0 - 1$ and yd, instead of $[s_0]$ and $[yd]$ respectively. This yields the lower bound

$$M_d\mu^d(x) \ge \frac{\left(1 - \frac{t}{\sqrt{d(2u)(1-2u)} + 2ut - 2t}\right)^{2ud - t\sqrt{d(2u)(1-2u)}}}{\left(1 - \frac{ut\sqrt{2}}{\sqrt{du(1-2u)} + ut\sqrt{2} - t\sqrt{2}}\right)^d}. \tag{9}$$

Next, we estimate the right hand side of the preceding expression, using a Taylor polynomial expansion about 0. Save d (which approaches ∞) everything else is held fixed, so we make the change of variables $z = t/\sqrt{d}$ and set

$$x(z) = \frac{z}{\sqrt{2u(1-2u)} + 2uz - 2z}, \tag{10}$$

$$y(z) = 2u - z\sqrt{2u(1-2u)}. \tag{11}$$

Defining

$$h(z) = \frac{(1-x(z))^{y(z)}}{1-2ux(z)}, \tag{12}$$

a computation (of which some details are presented below for the reader's convenience) shows that

$$h(z) = 1 + \frac{z^2}{2} + O(z^3). \tag{13}$$

Given (13), we obtain, for any $\varepsilon_0 > 0$ and every d large enough,

$$M_d\mu^d(x) \geq \left(1 + \frac{t^2}{2d} + O\left(\frac{t^3}{d^{3/2}}\right)\right)^d \geq e^{-\varepsilon_0} e^{t^2/2}. \tag{14}$$

We include next the promised details on the Taylor polymonial computation. Observe that $x(0) = 0$, $h(0) = 1$, $y(0) = 2u$, $x'(0) = 1/\sqrt{2u(1-2u)}$, and $y'(0) = -\sqrt{2u(1-2u)}$. Differentiating $h(z)$ we obtain

$$h'(z) = h(z)\left(y'(z)\log(1 - x(z)) + \frac{2ux'(z)}{1 - 2ux(z)} - \frac{x'(z)y(z)}{1 - x(z)}\right), \tag{15}$$

so $h'(0) = 0$, and

$$h''(z)|_{z=0} = \frac{d}{dz}\left(y'(z)\log(1 - x(z)) + \frac{2ux'(z)}{1 - 2ux(z)} - \frac{x'(z)y(z)}{1 - x(z)}\right)\Big|_{z=0} = 1. \tag{16}$$

Now (13) follows. $\square$

Next we estimate the size of the block E^u. It will be useful to recall some well known bounds for the tails of the standard normal distribution, which can be obtained via integration by parts (cf., for instance, [ClDr, pp. 111-112]): For all $x > 0$,

$$\frac{1}{x\sqrt{2\pi}}\left(1 - \frac{1}{x^2}\right)e^{-x^2/2} < \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt < \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}. \tag{17}$$

We will use (17) with $x >> 1$.

Claim 2: For all $u \in (0, 1/4]$, all $\varepsilon_1 \in (0, 1/10)$, and all d large enough (depending on u and ε_1) we have

$$\frac{1 - \varepsilon_1}{t\sqrt{2\pi}}\left(1 - \frac{2}{t^2}\right)e^{-t^2/2} < |E^u| < |E_u| < \frac{1 + \varepsilon_1}{t\sqrt{2\pi}}e^{-t^2/2}. \tag{18}$$

Proof. Consider the i-th coordinate of a point in the unit cube $[0,1]^d$ as a Bernoulli random variable $X_{u,i}$, off center with probability $2u$. Thus, the probability of having exactly k off center and $d - k$ centered coordinates is

$$\binom{d}{k}(2u)^k(1 - 2u)^{d-k}.$$

Set $S_{u,d} := \sum_{i=1}^d X_{u,i}$. Then $S_{u,d} \sim B(2u, d)$ is binomially distributed with mean $E(S_{u,d}) = 2ud$ and standard deviation $\sigma(S_{u,d}) = \sqrt{d(2u)(1-2u)}$. Furthermore, the Lebesgue measures of E_u and E^u are given, respectively, by $|E_u| = P\left(S_{u,d} \leq 2ud - t\sqrt{d(2u)(1-2u)}\right)$ and $|E^u| = |E_u| - P\left(S_{u,d} \leq 2ud - (t + t^{-1}4\log t)\sqrt{d(2u)(1-2u)}\right)$. Let $Z \sim N(0,1)$ and set

$v_1 = t$, $v_2 = t + t^{-1} 4 \log t$. By the Central Limit Theorem, given $\varepsilon_1 > 0$, for $i = 1, 2$ and all d large enough we have

$$(1 - \varepsilon_1) P(Z \leq -v_i) \leq P\left(S_{u,d} \leq 2ud - v_i \sqrt{d(2u)(1-2u)}\right) \leq (1 + \varepsilon_1) P(Z \leq -v_i). \tag{19}$$

Since $P(Z \leq -v_i) = \frac{1}{\sqrt{2\pi}} \int_{v_i}^{\infty} e^{-x^2/2} dx$ and $t^4 > t^2$ for all large t, (18) follows from (17). $\square$

Let $0 < v < u \leq 1/4$. Next we bound $|E^v \cap E^u|$, or equivalently, the relative size of E^v in E^u, under the additional hypothesis that u and v are far apart. Recall that $Z \sim N(0,1)$.

Claim 3: Fix $u \in (0, 1/4]$ and $v \in (0, u - t^{-2} 4 \log t)$. Then for all $\varepsilon_2 \in (0, 1/10)$ and all d large enough, we have

$$|E^v \cap E^u| \leq (1 + \varepsilon_2) P\left(Z \leq \frac{4 \log t \sqrt{v - 2vu}}{t\sqrt{u - v}} - t\sqrt{u - v}\right) \frac{1 + \varepsilon_1}{t\sqrt{2\pi}} e^{-t^2/2}, \tag{20}$$

where ε_1 is the error, coming from Claim 2, in the normal approximation.

Proof. Let us fix k with

$$2ud - (t + t^{-1} 4 \log t) \sqrt{d(2u)(1-2u)} < k \leq 2ud - t\sqrt{d(2u)(1-2u)}. \tag{21}$$

The number k counts how many off center coordinates can a point in E^u have. Within the corresponding $d-k$ set $A_{u,k}$, select a $d-k$ face $F_{u,k}$. Without loss of generality we may assume that $F_{u,k}$ has $x_1, \ldots, x_k$ as its k off center coordinates, and furthermore, that $x_i \in [0, u)$ for $i = 1, \ldots, k$, i.e., that the off center coordinates of $x \in F_{u,k}$ are all near 0. Thus, we can write $F_{u,k} = \prod_{i=1}^{k} [0, u) \times \prod_{i=k+1}^{d} [u, 1-u]$. We bound $P(E^v | F_{u,k}) = P(E^v \cap F_{u,k}) / P(F_{u,k})$.

Let us consider next the block defined by v. Fix a number m counting how many off center coordinates can a point $x \in E^v$ have. Hence,

$$2vd - (t + t^{-1} 4 \log t) \sqrt{d(2v)(1-2v)} < m \leq 2vd - t\sqrt{d(2v)(1-2v)}. \tag{22}$$

Within the corresponding $d - m$ set $A_{v,m}$, select a $d - m$ face $F_{v,m}$. From the fact that $v < u$ we obtain $2vd < 2ud - (t + t^{-1} 4 \log t)\sqrt{d(2u)(1-2u)}$, since d is "large enough". Thus, $m < k$. Without loss of generality we may assume, first, that the m off center coordinates of $F_{v,m}$ are among $x_1 \ldots, x_k$, for otherwise $F_{v,m} \cap F_{u,k} = \emptyset$. Second, that $F_{v,m}$ has its m off center coordinates near zero, for otherwise we again have $F_{v,m} \cap F_{u,k} = \emptyset$. And third, that the off center coordinates are precisely $x_1 \ldots, x_m$; this can be assumed by using symmetry considerations. Thus, $F_{v,m} = \prod_{i=1}^{m} [0, v) \times \prod_{i=m+1}^{d} [v, 1-v]$, and now we have

$$\frac{P(F_{v,m} \cap F_{u,k})}{P(F_{u,k})} = \prod_{i=1}^{m} \left(\frac{v}{u}\right) \times \prod_{i=m+1}^{k} \left(\frac{u-v}{u}\right) \times \prod_{i=k+1}^{d} \left(\frac{1-2u}{1-2u}\right) = \left(\frac{v}{u}\right)^m \left(1 - \frac{v}{u}\right)^{k-m}. \tag{23}$$

Since the number of $d - m$ faces in $A_{v,m}$ having nonempty intersection with $F_{u,k}$ is $\binom{k}{m}$, it follows from (23), by summing over all the $d - m$ faces of $A_{v,m}$, that

$$\frac{P(A_{v,m} \cap F_{u,k})}{P(F_{u,k})} = \binom{k}{m} \left(\frac{v}{u}\right)^m \left(1 - \frac{v}{u}\right)^{k-m}. \tag{24}$$

Thus,

$$P(E^v | F_{u,k}) = \sum_{m=1+\left[2vd-(t+t^{-1}4\log t)\sqrt{d(2v)(1-2v)}\right]}^{2vd-t\sqrt{d(2v)(1-2v)}} P(A_{v,m}|F_{u,k}) \tag{25}$$

$$= P(2vd - (t + t^{-1}4\log t)\sqrt{d(2v)(1-2v)} < S_{v/u,k} \le 2vd - t\sqrt{d(2v)(1-2v)}), \tag{26}$$

where $S_{v/u,k} \sim B(v/u, k)$. Once more we apply the Central Limit Theorem. Given $\varepsilon_2 \in (0, 1/10)$, for all $y \in \mathbb{R}$ there exists a $K > 0$ such that if $k \ge K$ (in short, if k, or equivalently d, is large enough) we have

$$(1 - \varepsilon_2)P(Z \le y) \le P\left(\frac{S_{v/u,k} - E(S_{v/u,k})}{\sigma(S_{v/u,k})} \le y\right) \le (1 + \varepsilon_2)P(Z \le y). \tag{27}$$

Now $E(S_{v/u,k}) = k\left(\frac{v}{u}\right)$, so from (21) we get

$$2vd - (t + t^{-1}4\log t)\sqrt{d\,(2v)\left(\frac{v}{u} - 2v\right)} < E(S_{v/u,k}) \le 2vd - t\sqrt{d\,(2v)\left(\frac{v}{u} - 2v\right)}. \tag{28}$$

Since

$$\sigma(S_{v/u,k}) = \sqrt{k\left(\frac{v}{u}\right)\left(1 - \frac{v}{u}\right)}, \tag{29}$$

from $k \le 2ud$ we obtain

$$\sigma(S_{v/u,k}) \le \sqrt{2vd\left(1 - \frac{v}{u}\right)}. \tag{30}$$

Set $w = 2vd - t\sqrt{d(2v)(1-2v)}$. Then (28) implies that

$$w - E(S_{v/u,k}) \le -t\sqrt{d(2v)(1-2v)} + (t + t^{-1}4\log t)\sqrt{d\,(2v)\left(\frac{v}{u} - 2v\right)} \tag{31}$$

$$= t\sqrt{2vd}\left(\left(1 + \frac{4\log t}{t^2}\right)\sqrt{\frac{v}{u} - 2v} - \sqrt{1-2v}\right) < 0, \tag{32}$$

where the last inequality follows from the hypothesis $u - v > t^{-2}4\log t$, by noticing that

$$\left(\frac{4\log t}{t^2}\right)\sqrt{\frac{v}{u} - 2v} - \frac{1 - \frac{v}{u}}{\sqrt{\frac{v}{u} - 2v} + \sqrt{1-2v}} < \frac{4\log t}{t^2} - \frac{4\log t}{2ut^2} < 0.$$

Thus, from (31), (32) and (30) we get

$$\frac{w - E(S_{v/u,k})}{\sigma(S_{v/u,k})} \leq \frac{t\left(\left(1+\frac{4\log t}{t^2}\right)\sqrt{\frac{v}{u}-2v} - \sqrt{1-2v}\right)}{\sqrt{1-\frac{v}{u}}} \tag{33}$$

$$= \frac{t\left(\frac{4\log t}{t^2}\right)\sqrt{\frac{v}{u}-2v}}{\sqrt{1-\frac{v}{u}}} - \frac{t\left(1-\frac{v}{u}\right)}{\left(\sqrt{\frac{v}{u}-2v}+\sqrt{1-2v}\right)\sqrt{1-\frac{v}{u}}} \leq \frac{4\log t\sqrt{v-2vu}}{t\sqrt{u-v}} - t\sqrt{u-v}, \tag{34}$$

so from the Central Limit Theorem we obtain

$$P(E^v \cap F_{u,k}) = P(E^v|F_{u,k})P(F_{u,k}) < (1+\varepsilon_2)P\left(Z \leq \frac{4\log t\sqrt{v-2vu}}{t\sqrt{u-v}} - t\sqrt{u-v}\right)P(F_{u,k}). \tag{35}$$

Next we use $F_{u,k}$ to denote a generic $d-k$ face in $A_{u,k}$. Since for $k \neq j$, $A_{u,k} \cap A_{u,j} = \emptyset$, and different $d-k$ faces within the same $d-k$ set are also disjoint, $|E^v \cap A_{u,k}| = \sum |E^v \cap F_{u,k}|$, where the sum is taken over all the $d-k$ faces in $A_{u,k}$, and $|E^v \cap E^u| = \sum_k |E^v \cap A_{u,k}|$, where the sum is taken over all the k's in the range given by (21). Thus, summing over all such k and all $F_{u,k}$ in $A_{u,k}$ we obtain $|E^u| = \sum P(F_{u,k})$ and

$$|E^v \cap E^u| = \sum P(E^v|F_{u,k})P(F_{u,k}) < (1+\varepsilon_2)P\left(Z \leq \frac{4\log t\sqrt{v-2vu}}{t\sqrt{u-v}} - t\sqrt{u-v}\right)\sum P(F_{u,k})$$

$$< (1+\varepsilon_2)P\left(Z \leq \frac{4\log t\sqrt{v-2vu}}{t\sqrt{u-v}} - t\sqrt{u-v}\right)\frac{1+\varepsilon_1}{t\sqrt{2\pi}}e^{-t^2/2},$$

since $|E^u| < \frac{1+\varepsilon_1}{t\sqrt{2\pi}}e^{-t^2/2}$ by Claim 2. □

Next we let u take the values $t^{-4/3}, 2t^{-4/3}, 3t^{-4/3} \dots \leq 1/4$.

Claim 4: Let $j = 1, 2, \dots$ *range over all the positive integers satisfying* $jt^{-4/3} \leq 1/4$, *and call* M *the largest such* j. *Then*

$$\left|\bigcup_{j=1}^{M} E^{jt^{-4/3}}\right| > \frac{t^{1/3}}{8\sqrt{\pi}}e^{-t^2/2}. \tag{36}$$

Proof. We use Claim 3 with $v = kt^{-4/3} < u = jt^{-4/3}$, where k and j are positive integers in $[1, 4^{-1}t^{4/3}]$. Since $u - v \geq t^{-4/3}$, using $t >> 1$ and $u, v \leq 1/4$ we conclude that

$$\frac{4\log t\sqrt{v-2vu}}{t\sqrt{u-v}} - t\sqrt{u-v} \leq \frac{2\log t}{t^{1/3}} - t^{1/3} < -\frac{t^{1/3}}{2}.$$

Now

$$P\left(Z \leq -\frac{t^{1/3}}{2}\right) < \frac{\sqrt{2}}{t^{1/3}\sqrt{\pi}}e^{-t^{2/3}/8},$$

so by (20) we have

(37)
$$\sum_{1\le k<j\le M}\left|E^{kt^{-4/3}}\cap E^{jt^{-4/3}}\right| < t^{8/3}\left(\frac{e^{-t^{2/3}/8}}{t^{1/3}}\right)\left(\frac{e^{-t^2/2}}{t}\right) = t^{4/3}e^{-t^{2/3}/8}e^{-t^2/2} = O\left(t^{-1}e^{-t^2/2}\right).$$

Using the inclusion exclusion principle, together with (37) and the lower bound from (18), we obtain

$$\left|\bigcup_{j=1}^{M} E^{jt^{-4/3}}\right| \ge \sum_{j=1}^{M}\left|E^{jt^{-4/3}}\right| - \sum_{1\le k<j\le M}\left|E^{kt^{-4/3}}\cap E^{jt^{-4/3}}\right| \tag{38}$$

$$> \left(\frac{t^{4/3}}{4}-1\right)\frac{(1-\varepsilon_1)}{t\sqrt{2\pi}}e^{-t^2/2} - O\left(t^{-1}e^{-t^2/2}\right) > \frac{t^{1/3}e^{-t^2/2}}{8\sqrt{\pi}}. \tag{39}$$

□

Completing the argument. The last step consists in fixing d (so large that all the preceding estimates hold) and replacing the infinite measure μ^d by a finite measure μ^d_R, such that the ratio of unit volume cubes and Dirac deltas is close to 1. The measure μ^d_R is obtained by keeping only the point masses of μ^d contained in the cube $\left[-\sqrt{d}, R+\sqrt{d}\right]^d$. This part of the proof (save for a small modification) already appears in [MS, Theorem 6].

Let f be an integrable function and ν a finite sum of Dirac deltas. By discretization, any lower bound for c in $\alpha\left|\{Mf\ge\alpha\}\right| \le c\|f\|_1$ is a lower bound for C in $\alpha\left|\{M\nu\ge\alpha\}\right| \le C\nu(\mathbb{R}^d)$, and viceversa. Here $\nu(\mathbb{R}^d)$ simply counts the number of point masses in ν. Observe that the cubes used in Claim 1 to estimate the size of $M\mu^d(x)$, for $x\in[0,1]^d$, never exceed a sidelength of $2\sqrt{d}$. Let $\mu^d_R := \sum_i \delta_{x_i}$, where $R = R(d) >> d$ is a natural number and $x_i\in\mathbb{Z}^d$ ranges over all the points with integer coordinates in the cube $[-\sqrt{d}, R+\sqrt{d}]^d$. Using the fact that the estimates in the preceding claims hold for every unit subcube of $[0,R]^d$ with vertices in $\mathbb{Z}^d$ (by the same argument presented for $[0,1]^d$) we have

$$c_d \ge \sup_{R>0}\frac{e^{-\varepsilon_0}e^{t^2/2}\left|\{M_d\mu^d_R > e^{-\varepsilon_0}e^{t^2/2}\}\cap[0,R]^d\right|}{\mu^d_R(\mathbb{R}^d)}$$
$$\ge \sup_{R>0}\left(\frac{R^d}{(R+2\sqrt{d}+1)^d}\right)\left(\frac{t^{1/3}e^{-t^2/2}e^{-\varepsilon_0}e^{t^2/2}}{8\sqrt{\pi}}\right) > \frac{t^{1/3}}{8\pi}$$

by taking ε_0 small enough. □

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